

NEW EXAMPLES OF PROPER HOLOMORPHIC MAPS AMONG SYMMETRIC DOMAINS

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1. INTRODUCTION

Let $\Omega_{r,s}$ be a bounded symmetric domain of type I which is defined by

$$\Omega_{r,s} = \{Z \in M(r, s, \mathbb{C}) : I_{r,r} - ZZ^* > 0\}.$$

Here we denote by > 0 positive definiteness of square matrices, by $M(r, s, \mathbb{C})$ the set of $r \times s$ complex matrices and by $I_{r,r}$ the $r \times r$ identity matrix. Let $D_{r,s}$ be a generalized ball which is defined by

$$D_{r,s} = \{[z_1, \dots, z_{r+s}] \in \mathbb{P}^{r+s-1} : |z_1|^2 + \dots + |z_r|^2 > |z_{r+1}|^2 + \dots + |z_{r+s}|^2\}.$$

Definition 1.1. (1) Let $f, g : \Omega_1 \rightarrow \Omega_2$ be holomorphic maps between domains Ω_1, Ω_2 . We say f and g are equivalent if and only if $f = A \circ g \circ B$ for some $B \in \text{Aut}(\Omega_1)$ and $A \in \text{Aut}(\Omega_2)$.
(2) Let $g_1, g_2 : \mathbb{P}^n \rightarrow \mathbb{P}^N$ be rational maps. We say g_1 and g_2 are rationally equivalent if there is a rational map $g : \mathbb{P}^n \rightarrow \mathbb{P}^N$ such that g is a common extension of g_1 and g_2 .

The aim of this paper is presenting a simple way to generate proper monomial rational maps between generalized balls and via the relations between generalized balls and bounded symmetric domains of type I given in [5], giving new examples of proper holomorphic maps between bounded symmetric domains of type I.

Consider a proper rational map $g : D_{r,s} \rightarrow D_{r',s'}$. In homogeneous coordinate, put $g([z_1, \dots, z_{r+s}]) = [g_1, \dots, g_{r'+s'}]$. Suppose that g_i are monomials in z_1, \dots, z_{r+s} for each i , $1 \leq i \leq r' + s'$. Then we can define the homogeneous polynomial $P : \mathbb{R}^{r+s} \rightarrow \mathbb{R}$ satisfying

$$(1.1) \quad P(|z_1|^2, \dots, |z_{r+s}|^2) = \sum_{k=1}^{r'} |g_k|^2 - \sum_{k=r'+1}^{r'+s'} |g_k|^2.$$

Since g is proper, $P(x) = 0$ whenever $\sum_{j=1}^r x_j = \sum_{j=r+1}^{r+s} x_j$. Hence P should be of the form

$$(1.2) \quad \left(\sum_{j=1}^r x_j - \sum_{j=r+1}^{r+s} x_j \right)^m Q_P(x)$$

for some positive integer m and homogeneous polynomial $Q_P(x)$.

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Theorem 1.2. *Let $g : D_{r,r} \rightarrow D_{r+1,r+1}$, ($r \geq 2$) be a proper monomial rational map. Then g is rationally equivalent to one of the following up to automorphisms of $D_{r,r}$ and $D_{r+1,r+1}$:*

- (1) *In case of $\deg(g) = 1$:*
 $g([z_1, \dots, z_{2r}]) = [z_1, \dots, z_r, \phi(z), z_{r+1}, \dots, z_{2r}, \phi(z)]$ where $\phi(z)$ is a degree one homogeneous polynomial in z_1, \dots, z_{2r}
- (2) *In case of $\deg(g) = 2$:*
 - (a) $g([z_1, z_2, z_3, z_4]) = [z_1^2, z_1 z_2, z_2 z_3, z_3^2, z_1 z_4, z_3 z_4]$
 - (b) $g([z_1, z_2, z_3, z_4]) = [z_1^2, \sqrt{2} z_1 z_2, z_2^2, z_3^2, \sqrt{2} z_3 z_4, z_4^2]$
- (3) *In case of $\deg(g) \geq 3$: if $Q_P(x)$ has degree 1 or, the coefficients of the polynomial $Q_P(x)$ are nonnegative, there is no proper monomial rational map.*

The condition in Theorem 1.2 about Q_P are due to combinatorial method counting monomials in expansion of multiplied polynomial.

The method to characterize proper monomial rational maps originally comes from J. P. D'Angelo in [1]. He studied proper monomial holomorphic maps from the unit ball to the higher dimensional unit ball via characterizing the polynomials which can be obtained by taking Euclidean norm on proper maps. By characterizing these polynomials, he obtained complete list of proper monomial holomorphic maps from the two dimensional unit ball to the four dimensional unit ball. In this paper, we modify this polynomial which is appropriate to proper monomial rational maps between generalized balls and characterize the polynomial by counting the number of monomials in the polynomial.

For bounded symmetric domains of rank at least 2, properties of proper holomorphic maps are deeply related to special kind of totally geodesic subspaces of given domains which are called *invariantly geodesic subspaces*. These are totally geodesic submanifolds with respect to the Bergman metric which are still totally geodesic under the action of automorphisms of the compact dual of ambient domain. Invariantly geodesic subspaces first appeared in [3] as far as the author knows. These subspaces play important roles to characterize proper holomorphic maps between bounded symmetric domains. Especially N. Mok and I. H. Tsai proved that proper holomorphic maps between irreducible bounded symmetric domains preserve the maximal characteristic subspaces which are also invariantly geodesic subspaces. Based on [3, 7], the rigidity of irreducible bounded symmetric domains have been developed and incorporated by Z. Tu [8, 9] and S. C. Ng [5, 6]. Especially, Ng [5] found that generalized balls in the projective spaces parametrize the maximal invariantly geodesic subspaces of bounded symmetric domains of type I and we use this relation to find several examples of proper holomorphic maps between bounded symmetric domains of type I.

Consider the subspaces in $\Omega_{r,s}$ of the form

$$L_{[A,B]} = \{Z \in \Omega_{r,s} : AZ = B\}$$

where $A \in M(1, r, \mathbb{C})$, $b \in M(1, s, \mathbb{C})$ satisfying $[A, B] \in D_{r,s}$ which are totally geodesic under the action of $SL(r+s, \mathbb{C})$. These are the maximal invariantly geodesic subspaces. For $X = [A, B] \in D_{r,s}$, denote $X^\# = L_X$.

For a proper holomorphic map $f : D_{r,r} \rightarrow D_{r+1,r+1}$, ($r \geq 2$) which preserves the maximal invariantly geodesic subspaces, there is a proper holomorphic map $g : D_{r,r} \rightarrow D_{r+1,r+1}$ such that $f(X^\#) \subset g(X)^\#$ for generic $X \in D_{r,r}$.

Theorem 1.3. *Let $f : \Omega_{r,r} \rightarrow \Omega_{r+1,r+1}$, ($r \geq 2$) be a proper holomorphic map. Suppose that f preserves the maximal invariantly geodesic subspaces and an induced proper holomorphic map $g : D_{r,r} \rightarrow D_{r+1,r+1}$ satisfies the conditions in Theorem 1.2. Then f is equivalent to one of the following:*

- (1) $f(Z) = \begin{pmatrix} Z & 0 \\ 0 & h(Z) \end{pmatrix}$ for $Z \in \Omega_{r,r}$
and for some holomorphic map $h : \Omega_{r,r} \rightarrow \Delta = \{z \in \mathbb{C} : |z| < 1\}$.
- (2) $f\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\right) = \begin{pmatrix} z_1^2 & z_1 z_2 & z_2 \\ z_1 z_3 & z_2 z_3 & z_4 \\ z_3 & z_4 & 0 \end{pmatrix}$, for $\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \Omega_{2,2}$
- (3) $f\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\right) = \begin{pmatrix} z_1^2 & \sqrt{2}z_1 z_2 & z_2^2 \\ \sqrt{2}z_1 z_3 & z_1 z_4 + z_2 z_3 & \sqrt{2}z_2 z_4 \\ z_3^2 & \sqrt{2}z_3 z_4 & z_4^2 \end{pmatrix}$

Here is the outline of the paper. Section 2 introduces some basic terminology, well-known facts and the invariantly geodesic subspaces. In section 3, we modify D'Angelo's method to proper monomial maps between generalized balls and classify the maps which are needed to sort proper holomorphic maps between bounded symmetric domains of type I. We count the number of monomials in homogeneous polynomial which is multiplied by two homogeneous polynomials. In Section 4, we present a way to generate proper holomorphic maps from $\Omega_{r,s}$ to $\Omega_{r',s'}$ and prove Theorem 1.3. Furthermore we give more examples which are interesting.

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2. PRELIMINARY

2.1. Basic facts and Terminology. At first, we introduce terminology and some facts. For more detail, see [5, 3]. Let $G_{r,s}$ be the Grassmannian of r -planes in $r+s$ dimensional complex vector space \mathbb{C}^{r+s} which is the compact dual of $\Omega_{r,s}$. For $X \in M(r, r+s, \mathbb{C})$ of rank r , denote $[X]$ an r -plane in \mathbb{C}^{r+s} which is generated by row vectors of X . For each element Z in $\Omega_{r,s}$, there corresponds an r -plane $[I_{r,r}, Z] \in G_{r,s}$. This is the Borel embedding of $\Omega_{r,s}$ into $G_{r,s}$. It is clear that $SL(r+s, \mathbb{C})$ acts holomorphically and transitively $G_{r,s}$. Denote $SU(r, s)$ the subgroup of $SL(r+s, \mathbb{C})$ satisfying $M \begin{pmatrix} -I_{r,r} & 0 \\ 0 & I_{s,s} \end{pmatrix} M^* = \begin{pmatrix} -I_{r,r} & 0 \\ 0 & I_{s,s} \end{pmatrix}$ for all $M \in SU(r, s)$. Then

$SU(r, s)$ is the automorphism group of $\Omega_{r,s}$. If we put $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A \in M(r, r, \mathbb{C})$, $B \in M(r, s, \mathbb{C})$, $C \in M(s, r, \mathbb{C})$, $D \in M(s, s, \mathbb{C})$, M acts on $\Omega_{r,s}$ by $Z \mapsto (A + ZC)^{-1}(B + ZD)$. From now on, if we write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(r, s)$, without ambiguity, A, B, C, D are block matrices of the above form.

2.2. Invariantly geodesic subspaces in $\Omega_{r,s}$. Consider a complex submanifold S in $\Omega_{r,s}$. For every $g \in SL(r+s, \mathbb{C})$ such that $g(S) \cap \Omega_{r,s} \neq \emptyset$, if the submanifold

$g(S) \cap \Omega_{r,s}$ is totally geodesic in $\Omega_{r,s}$ with respect to the Bergman metric of $\Omega_{r,s}$, then S is called *invariantly geodesic subspace* of $\Omega_{r,s}$. In particular, for $W \in \Omega_{r',s'}$ with $r' \leq r$ and $s' \leq s$, the image of the embedding $i : W \mapsto \begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix} \in \Omega_{r,s}$ is an invariantly geodesic subspace of $\Omega_{r,s}$. The totally geodesic subspaces which are equivalent under the action of $SU(r, s)$ to $i(\Omega_{r',s'})$ in $\Omega_{r,s}$ are called (r', s') -subspaces of $\Omega_{r,s}$. Among these (r', s') -subspaces, the maximal invariantly geodesic subspaces are parametrized by the generalized ball in \mathbb{P}^{r+s-1} .

Proposition 2.1 ([5]). *The subspaces of the form*

$$(2.1) \quad L = \{Z \in \Omega_{r,s} : AZ = B\}$$

where $A \in M(1, r, \mathbb{C})$, $B \in M(1, s, \mathbb{C})$ satisfying $[A, B] \in D_{r,s}$ are $(r-1, s)$ -subspaces.

For example, in case of invariantly geodesic subspaces

$$\left\{ \begin{pmatrix} 0 \\ W \end{pmatrix} \in \Omega_{r,s} : W \in \Omega_{r-1,s} \right\},$$

$A = (1, 0, \dots, 0) \in M(1, r, \mathbb{C})$ and $B = (0, \dots, 0) \in M(1, s, \mathbb{C})$.

For $\Omega_{r,s}$ and $D_{r,s}$, consider the two surjective maps

$$(2.2) \quad \phi : \mathbb{P}^{r-1} \times \Omega_{r,s} \rightarrow \Omega_{r,s}, \quad ([X], Z) \mapsto Z$$

$$(2.3) \quad \psi : \mathbb{P}^{r-1} \times \Omega_{r,s} \rightarrow D_{r,s}, \quad ([X], Z) \mapsto [X, XZ].$$

For $Z \in \Omega_{r,s}$, denote $Z^\# = \psi(\phi^{-1}(Z)) \subset D_{r,s}$. Similarly for $X \in D_{r,s}$, denote $X^\# = \phi(\psi^{-1}(X)) \subset \Omega_{r,s}$. $Z^\#$ and $X^\#$ are called fibral images of Z and X respectively. Then for $Z \in \Omega_{r,s}$ and $X = [A, B] \in D_{r,s}$ where $A \in M(1, r, \mathbb{C})$ and $B \in M(1, s, \mathbb{C})$,

$$(2.4) \quad Z^\# = \{[A, AZ] \in D_{r,s} : [A] \in \mathbb{P}^{r-1}\} \cong \mathbb{P}^{r-1}$$

$$(2.5) \quad X^\# = \{Z \in \Omega_{r,s} : AZ = B\} \cong (r-1, s)\text{-subspace}$$

Proposition 2.2 (cf. [5]). *Let $f : \Omega_{r,r} \rightarrow \Omega_{r+1,r+1}$ be a proper holomorphic map. Suppose that there is a meromorphic map $g : D_{r,r} \rightarrow D_{r+1,r+1}$ such that $f(X^\#) \subset g(X)^\#$ for generic point $X \in D_{r,r}$. Then g is a proper map or $f(Z) = \begin{pmatrix} Z & 0 \\ 0 & h(Z) \end{pmatrix}$ for some holomorphic function $h : \Omega_{r,r} \rightarrow \Delta$.*

3. PROPER MONOMIAL RATIONAL MAP FROM $D_{r,s}$ TO $D_{r',s'}$

Let $g : D_{r,s} \rightarrow D_{r',s'}$ be a proper monomial rational map and P, Q_P be homogeneous polynomials defined by (1.1) and (1.2). Then Q_P has the following properties:

(1) $Q_P(x)$ is a homogeneous polynomial which is not identically zero on

$$\left\{ x = \{x_1, \dots, x_{r+s}\} \in \mathbb{R}^{r+s} : \sum_{j=1}^r x_j = \sum_{j=r+1}^{r+s} x_j \right\}$$

(2) $Q_P(x) > 0$ whenever $x_i > 0$ for all i and $\sum_{j=1}^r x_j > \sum_{j=r+1}^{r+s} x_j$.

3.1. Classifying proper monomial rational map from $D_{r,r}$ to $D_{r+1,r+1}$. A situation of classifying proper rational maps between generalized balls is different from that of classifying proper holomorphic maps between unit balls in [2] since there are infinite number of proper rational maps which are same in open dense subset. For example, $g : D_{2,2} \rightarrow D_{3,3}$, $[z_1, \dots, z_4] \mapsto [z_1 h, z_2 h, 0, z_3 h, z_4 h, 0]$ for any holomorphic function h of \mathbb{C}^4 which is not identically zero on $D_{2,2}$ are same in open dense subset depending on zero set of h . On the other hand, proper rational maps which are same in open dense subset induce the same proper holomorphic map between corresponding bounded symmetric domains of type I. Hence we consider equivalence relation on proper monomial rational maps to incorporate these infinite number of rational maps.

Definition 3.1. Let $g_1, g_2 : \mathbb{P}^{2r-1} \rightarrow \mathbb{P}^{2r+1}$ be rational maps. We say g_1 and g_2 are rationally equivalent if there is a rational map $g : \mathbb{P}^{2r-1} \rightarrow \mathbb{P}^{2r+1}$ such that g is a common extension of g_1 and g_2 .

We may assume that every components of $g : D_{r,s} \rightarrow D_{r',s'}$ have no common factor.

In the rest of this section, we characterize the induced polynomial $P(x)$ and the proper monomial rational maps from $D_{r,r}$ to $D_{r+1,r+1}$ to prove Theorem 1.2. For this aim, we will count the number of monomials of P for suitable Q_P . For a polynomial A , denote $n_i(A)$ the number of monomials with maximal degree in x_i of A and $n(A)$ the number of monomials in A .

Lemma 3.2. For polynomial $A = (b_1 x_1 + \dots + b_k x_k)^m \tilde{A}$ with nonzero polynomial \tilde{A} , positive integer m and nonzero b_i for all i , $1 \leq i \leq k$, $n(A) \geq \sum_{i=1}^k n_i(\tilde{A})$.

Proof. $(b_i x_i)^m$ times the monomial with the maximal degree of x_i in \tilde{A} cannot be canceled. \square

Lemma 3.3. Let $P(x)$ be a homogeneous polynomial on \mathbb{R}^k of the form

$$(b_1 x_1 + \dots + b_k x_k)^m Q_P(x)$$

for some positive integer m , nonzero b_i for all i , $1 \leq i \leq k$ and homogeneous polynomial $Q_P(x)$ with nonnegative coefficients. Then if $m \geq 2$, $n(P) \geq 2k - 1$.

Proof. Without loss of generality, we may assume that $Q_P(x)$ contain x_1 variable with $b_1 > 0$ and $n(Q_P) \geq 2$. Let $Q_P(x) = A_0 + A_1 x_1 + A_2 x_1^2 + \dots + A_\alpha x_1^\alpha$ be the expansion of $Q_P(x)$ with respect to the degree of x_1 variable where α is the maximal degree of x_1 in $Q_P(x)$, A_l is a homogeneous polynomial without x_1 variable having nonnegative coefficients and A_0 and A_α are nonzero. Denote $B = b_2 x_2 + \dots + b_k x_k$. Then

$$P(x) = A_0 B^m + x_1 B^{m-1} (m b_1 A_0 + A_1 B) + \dots + x_1^{\alpha+m} A_\alpha.$$

Note that there are at least $k - 1$ monomials in $A_0 B^m$ and 1 monomial in $x_1^{\alpha+m} A_\alpha$. Notice that the second term $x_1 B^{m-1} (m b_1 A_0 + A_1 B)$ is not vanish and has at least $k - 1$ monomials. Hence summing up, there are at least $2k - 1$ monomials in P when $m \geq 2$. \square

Lemma 3.4. Let $P(x)$ be a homogeneous polynomial on \mathbb{R}^{2r} of the form

$$(x_1 + \dots + x_r - x_{r+1} - \dots - x_{2r}) Q_P(x)$$

for some homogeneous polynomial $Q_P(x)$ with nonnegative coefficients and $n(Q_P) \geq 2$. Then

- (1) $n(P) \geq 3r - 1$ if $r \geq 2$,
- (2) $n(P) \geq 9$ if $r = 3$.

Proof. As in the proof of Lemma 3.3, consider

$$P(x) = A_0B + x_1(A_0 + A_1B) + x_1^2(A_1 + A_2B) + \cdots + A_\alpha x_1^{\alpha+1}.$$

Suppose $A_i = 0$ but $A_{i+1} \neq 0$ for some $i, 1 \leq i \leq \alpha - 1$. Then the coefficient of x_1^{i+1} is $A_{i+1}B$ and then there exist at least $2r - 1$ monomials which cannot be canceled. This implies that in this case, $n(P) \geq 4r - 1$. Hence it is enough to consider when $A_i \neq 0$ for any $i, 0 \leq i \leq \alpha$. In this case, there are at least $2r - 1$ monomials in A_0B , $r - 1$ monomials in $x_1(A_0 + A_1B)$, $r - 1$ monomials in $x_1^2(A_1 + A_2B)$ and 1 monomial in $A_\alpha x_1^{\alpha+1}$. Hence $n(P) \geq 3r - 1$.

Consider $r = 3$. We may assume that $A_i \neq 0$ for all i . Since $n(A_i + A_{i+1}) \geq 2$ for all i , it is enough to consider when $\alpha = 1$. Then $P(x) = A_0B + x_1(A_0 + A_1B) + A_1x_1^2$. If $A_0 = A_1(x_4 + x_5 + x_6)$, then $n(A_0B) \geq 9$ and if $A_0 \neq A_1(x_4 + x_5 + x_6)$, then $n(x_1(A_0 + A_1B)) \geq 3$. Hence $n(P) \geq 9$. \square

Lemma 3.5. Let $P(x)$ be a nonzero homogeneous polynomial on \mathbb{R}^k ($k \geq 1$) of the form

$$(b_1x_1 + \cdots + b_kx_k)^m(a_1x_1 + \cdots + a_kx_k)$$

for some positive integer m , $a_i \in \mathbb{R}$ for $i, 1 \leq i \leq k$ and nonzero b_i for all $i, 1 \leq i \leq k$. Then

- (1) if $m \geq 2$, then $n(P) \geq 2k - 1$
- (2) If $m = 1$ and $n(a_1x_1 + \cdots + a_kx_k) \geq 2$, then $n(P) \geq 2k - 2$.

Proof. We will prove (1). The proof of (2) is similar.

If $n(a_1x_1 + \cdots + a_kx_k) = 1$, then there are $\binom{k+m-1}{m} \geq 2k - 1$ number of monomials in P .

Suppose that $n(a_1x_1 + \cdots + a_kx_k) \geq 2$. We may assume that $a_1 \neq 0$. Put $A = a_2x_2 + \cdots + a_kx_k$ and $B = b_2x_2 + \cdots + b_kx_k$. Then

$$P(x) = B^m A + x_1 B^{m-1}(mb_1A + a_1B) + \cdots + a_1x_1^{m+1}.$$

Consider the case $mb_1A + a_1B \neq 0$. Let x be the number of a_i 's which are zero and y be the number of a_i 's which are nonzero. Then $n(B^m A) \geq y - 1 + x(y - 1) = -y^2 + (k + 2)y - 1 - k$ for $y, 2 \leq y \leq k$. At $y = 2$ the minimum $k - 1$ appears. Hence $n(P) \geq n(B^m A) + n(B^{m-1}(mb_1A + a_1B)) + n(a_1x_1^{m+1}) \geq 2k - 1$.

If $mb_1A + a_1B = 0$, $n(B^m A) = n(B^{m+1}) = \binom{k+m}{m} \geq 2k - 1$. \square

Lemma 3.6. Let $P(x) = (x_1 + x_2 - x_3 - x_4)Q_P(x)$ for $Q_P(x) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$, $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4$. Suppose that $n(P) \leq 6$ and

$$(3.1) \quad Q_P(x) > 0 \text{ whenever } x_1 + x_2 > x_3 + x_4 \text{ and } x_i > 0 \text{ for all } i, 1 \leq i \leq 4,$$

then the $Q_P(x)$ is one of the following up to multiplication of constants:

$$x_1, x_2, x_3, x_4, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_1 + x_2 + x_3 + x_4$$

Proof. We prove the lemma case by case.

$$(3.2) \quad \begin{aligned} P(x) = & a_1x_1^2 + a_2x_2^2 - a_3x_3^2 - a_4x_4^2 + (a_2 + a_1)x_1x_2 + (a_3 - a_1)x_1x_3 \\ & + (a_3 - a_2)x_2x_3 + (a_4 - a_1)x_1x_4 + (a_4 - a_2)x_2x_4 - (a_3 + a_4)x_3x_4 \end{aligned}$$

- (1) If only one a_i is zero and the others are nonzero, then Q_P is x_i for $1 \leq i \leq 4$.
- (2) If $a_1 = 0$ and $a_i \neq 0$ where $2 \leq i \leq 4$, then there are monomials, x_1x_i , x_i^2 for $2 \leq i \leq 4$ which cannot be canceled. Hence $a_2 = a_3$, $a_2 = a_4$, $a_4 + a_3 = 0$ and this is a contradiction. If $a_j = 0$ and $a_k \neq 0$ for $k \neq j$, by the same way, this cannot happen.
- (3) If $a_1 = a_2 = 0$, $a_3 \neq 0$, $a_4 \neq 0$, then $a_3 + a_4 = 0$. This contradicts to the condition (3.1). Similarly, there is no Q_P for $a_3 = a_4 = 0$, $a_1 \neq 0$, $a_2 \neq 0$.
- (4) If $a_2 = a_4 = 0$, $a_1 \neq 0$, $a_3 \neq 0$, then $a_1 = a_3$ and $a_1 > 0$. This case corresponds to $Q_P(x) = x_1 + x_3$ and similarly, cases, $\{a_1 = a_3 = 0, a_2 \neq 0, a_4 \neq 0\}$, $\{a_1 = a_4 = 0, a_3 \neq 0, a_2 \neq 0\}$, $\{a_3 = a_2 = 0, a_1 \neq 0, a_4 \neq 0\}$ corresponds to $x_2 + x_4$, $x_3 + x_2$, $x_1 + x_4$ respectively.
- (5) If all a_i are nonzero, by (3.1) $a_1 > 0$, $a_2 > 0$. Hence at least 3 monomial among $(a_3 - a_1)x_1x_3$, $(a_3 - a_2)x_2x_3$, $(a_4 - a_1)x_1x_4$, $(a_4 - a_2)x_2x_4$ should be zero. This implies that $a_1 = a_2 = a_3 = a_4$.

□

Proof of Theorem 1.2. Let

$$(x_1 + \cdots + x_r - x_{r+1} - \cdots - x_{2r})^m Q_P(x)$$

be the homogeneous polynomial induced by g for some positive integer m and homogeneous polynomial $Q_P(x)$. Then P satisfies $n(P) \leq 2r + 2$. If $n(Q_P) = 1$, g is rationally equivalent to (1). Hence we only need to consider when $n(Q_P) \geq 2$.

Suppose $m \geq 2$. Then by Lemma 3.3 and 3.5, $n(P) \geq 4r - 1 > 2r + 2$. Hence $m = 1$. On the other hand, by Lemma 3.5 and 3.4, $n(P) \geq 2r + 2$ for all $r \geq 3$.

For $m = 1$, $r = 2$, by Lemma 3.6,

$$\begin{aligned} & x_1 + x_2 - x_3 - x_4, x_1^2 + x_1x_2 + x_2x_3 - x_3^2 - x_1x_4 - x_3x_4, \\ & x_2^2 + x_1x_2 + x_1x_4 - x_4^2 - x_2x_3 - x_3x_4, x_1^2 + x_1x_2 + x_2x_4 - x_4^2 - x_1x_3 - x_3x_4, \\ & x_2^2 + x_1x_2 + x_1x_3 - x_3^2 - x_2x_4 - x_3x_4, x_1^2 + 2x_1x_2 + x_2^2 - x_3^2 - 2x_3x_4 - x_4^2. \end{aligned}$$

Then the first one induces (1) and the last one induce the map (2b). The second to fifth one induce the map equivalent to (2a). □

4. PROPER HOLOMORPHIC MAPS BETWEEN BOUNDED SYMMETRIC DOMAINS

4.1. Construction of proper holomorphic maps from $\Omega_{r,s}$ to $\Omega_{r',s'}$. In this section, using the relations between $(r-1, s)$ -subspaces in $\Omega_{r,s}$ and projective subspaces ($\cong \mathbb{P}^{r-1}$) in $D_{r,s}$ which is given in [5], we describe the construction of proper holomorphic mapping between bounded symmetric spaces of type I. To consider the boundary behavior of g , extend ϕ and ψ to

$$\begin{aligned} \tilde{\phi} : \mathbb{P}^{r-1} \times \overline{\Omega}_{r,s} &\rightarrow \overline{\Omega}_{r,s}, ([X], Z) \mapsto Z \\ \tilde{\psi} : \mathbb{P}^{r-1} \times \overline{\Omega}_{r,s} &\rightarrow \overline{D}_{r,s}, ([X], Z) \mapsto [X, XZ]. \end{aligned}$$

For the boundary points, consider the fibral image with respect to this extended map. Let $z \in \partial\Omega_{r,s}$. This implies that z satisfies $I_{r,r} - z\bar{z}^t \geq 0$ and there is $a \in \mathbb{C}^r$

such that $a(I_{r,r} - z\bar{z}^t)\bar{a}^t = 0$. Hence $z^\#$ may not be contained in $\partial D_{r,s}$ and

$$(4.1) \quad z^\# \cap \partial D_{r,s} = \{[a, az] \in \overline{D}_{r,s} : [a] \in \mathbb{P}^{r-1}, a(I_{r,r} - z\bar{z}^t)\bar{a}^t = 0\}$$

On the other hand, for $[a, b] \in \partial D_{r,s}$ where $a \in M(1, r, \mathbb{C})$ and $b \in M(1, s, \mathbb{C})$, if $z \in [a, b]^\#$, $a\bar{a}^t = b\bar{b}^t = az\overline{(az)}^t = az\bar{z}^t\bar{a}^t$. Hence for $[a, b] \in \partial D_{r,s}$, $[a, b]^\# \subset \partial \Omega_{r,s}$.

Definition 4.1. For a rational map $g : D_{r,s} \rightarrow D_{r',s'}$, we say rational map g is *proper* if for any point $x \in \partial D_{r,s}$ and open neighborhood U of x which does not intersect the indeterminacy of g , g is proper on $U \cap D_{r,s}$.

Proposition 4.2. Let $f : \Omega_{r,s} \rightarrow \Omega_{r',s'}$ be a holomorphic map. Suppose that there is a proper rational map $g : D_{r,s} \rightarrow D_{r',s'}$ satisfying

$$(4.2) \quad f(X^\#) \subset g(X)^\# \text{ for generic point } X \in D_{r,s}.$$

Then f is proper.

Proof. Let $\{Z_j\}$ be a sequence in $\Omega_{r,s}$ such that $Z_j \rightarrow z \in \partial \Omega_{r,s}$. Choose points $X_j \in Z_j^\#$ and $x \in \partial D_{r,s} \cap z^\#$ such that $X_j \rightarrow x$. Then since $g(X_j) \rightarrow g(x)$, $f(Z_j) \in f(X_j^\#) \subset g(X_j)^\# \rightarrow g(x)^\# \subset \partial \Omega_{r',s'}$. Hence f is proper. \square

Let $f : \Omega_{r,s} \rightarrow \Omega_{r',s'}$ be a proper holomorphic maps which is provided from a proper rational map $g : D_{r,s} \rightarrow D_{r',s'}$ satisfying the condition in Proposition 4.2. Denote $g = [g_1, g_2]$ where g_1 has r' -components and g_2 has s' -components. For $X = [A, B] \in D_{r,s}$ and $Z \in X^\#$ i.e. $B = AZ$. Then $f([A, AZ]^\#) \subset g([A, AZ]^\#)$ and this implies that

$$(4.3) \quad g_1([A, AZ])f(Z) = g_2([A, AZ]) \text{ for all } A \in \mathbb{P}^{r-1}.$$

Proposition 4.3. Let $g = [g_1, g_2] : D_{r,s} \rightarrow D_{r',s'}$ be a proper rational map. Let $f : M(r, s, \mathbb{C}) \rightarrow M(r', s', \mathbb{C})$ be a holomorphic map satisfying (4.3). Suppose that for generic points $Z \in \Omega_{r,s}$, there are r' points $\{X_i : 1 \leq i \leq r'\}$ in \mathbb{P}^{r-1} such that $\{g_1([X_i, X_i Z]) : 1 \leq i \leq r'\}$ are independent as r' vectors in $\mathbb{C}^{r'}$. Then $f(\Omega_{r,s}) \subset \Omega_{r',s'}$.

Proof. By (4.3), for $Z \in \Omega_{r,s}$

$$g_1([X_i, X_i Z]) (I_{r',r'} - f(Z)f(Z)^*) g_1([X_i, X_i Z])^* > 0.$$

Hence if $\{g_1([X_i, X_i Z]) : 1 \leq i \leq r'\}$ are independent, $I_{r',r'} - f(Z)f(Z)^*$ is positive definite. This implies the proposition. \square

Hence for a proper rational map g satisfying the condition in Proposition 4.3, if we find a solution of the system of equations (4.3), we get a proper holomorphic maps by Proposition 4.2.

Remark 4.4. For a meromorphic map $g : D_{r,s} \rightarrow D_{r',s'}$ and a holomorphic map $f : \Omega_{r,s} \rightarrow \Omega_{r',s'}$ satisfying (4.2), put g' a meromorphic map $h \circ g_2 \circ h'$ for some $h' \in \text{Aut}(D_{r,s})$ and $h \in \text{Aut}(D_{r',s'})$. Then there is $H \in \text{Aut}(\Omega_{r,s})$ and $H' \in \text{Aut}(\Omega_{r',s'})$ such that g' and $f' := H' \circ f \circ H$ satisfies (4.2). This is due to the construction of (2.2) and for more detail, see [5].

4.2. Proof of Theorem 1.3. Note that two rationally equivalent proper monomial rational maps from $D_{r,r}$ to $D_{r+1,r+1}$ induce the same proper holomorphic map from $\Omega_{r,r}$ to $\Omega_{r+1,r+1}$. By Theorem 1.2, there are three possibilities to be g . (2b) and (2a) satisfies the condition in Proposition 4.3. We will only induce the proper map (2a) since calculation of map (2b) is similar. Proper rational map is given by $g([x_1, x_2, x_3, x_4]) = [x_1^2, x_1x_2, x_2x_3, x_3^2, x_1x_4, x_3x_4]$. Let $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \Omega_{2,2}$. Then

$$\begin{aligned} Z^\# &= \{[x_1, x_2, x_1z_1 + x_2z_3, x_1z_2 + x_2z_4] \in D_{2,2} : [x_1, x_2] \in \mathbb{P}^1\} \\ g([x_1, x_2, x_1z_1 + x_2z_3, x_1z_2 + x_2z_4]) &= [A, B] \text{ where} \\ A &= (x_1^2, x_1x_2, x_2(x_1z_1 + x_2z_3)), \\ B &= ((x_1z_1 + x_2z_3)^2, x_1(x_1z_2 + x_2z_4), (x_1z_1 + x_2z_3)(x_1z_2 + x_2z_4)). \end{aligned}$$

Denote $f(Z) = \begin{pmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{pmatrix}$, then

$$\begin{aligned} x_1^2 L_1 + x_1 x_2 L_2 + x_2 (x_1 z_1 + x_2 z_3) L_3 &= (x_1 z_1 + x_2 z_3)^2 \\ x_1^2 M_1 + x_1 x_2 M_2 + x_2 (x_1 z_1 + x_2 z_3) M_3 &= x_1 (x_1 z_2 + x_2 z_4) \\ x_1^2 N_1 + x_1 x_2 N_2 + x_2 (x_1 z_1 + x_2 z_3) N_3 &= (x_1 z_1 + x_2 z_3)(x_1 z_2 + x_2 z_4) \end{aligned}$$

for all $[x_1, x_2] \in \mathbb{P}^1$. Hence we obtain (2).

Consider the case (1) in Theorem 1.2. Suppose that for simplicity suppose that $g : D_{2,2} \rightarrow D_{3,3}$ is $g(x) = x_1$. This method can be applied to general r and homogeneous monomial linear map g . The induce map $f : \Omega_{2,2} \rightarrow \Omega_{3,3}$ has the form

$$f\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\right) = \begin{pmatrix} z_1 - L & z_2 - M & 1 - N \\ z_3 & z_4 & 0 \\ L & M & N \end{pmatrix}$$

for some holomorphic functions L, M, N on $\Omega_{2,2}$. Since $I_{3,3} - f(Z)f(Z)^* > 0$, for $V = (v, 1, 0)$ and Z in the Shilov boundary of $\Omega_{2,2}$,

$$\begin{aligned} 0 &\leq V(I_{3,3} - f(Z)f(Z)^*)V^* \\ &= 1 + v^2 - |v(z_1 - L) + z_3|^2 - |v(z_2 - M) + z_4|^2 - |v(1 - N)|^2 \\ (4.4) \quad &= -v^2(|L|^2 + |M|^2 + |N|^2) + (\text{first order term in } v, \bar{v}) \end{aligned}$$

As $v \rightarrow \infty$, (4.4) tends to $-\infty$, if one of L, M, N are nonzero at Z . This implies that L, M, N should be zero on the Shilov boundary of $\Omega_{2,2}$ and hence $L = M = N = 0$ on $\Omega_{2,2}$. However in this case f is not a holomorphic map into $\Omega_{3,3}$. Thus there is no proper holomorphic map induced from g with nonzero ϕ .

If $g(x) = [x_1, x_2, 0, x_3, x_4, 0]$, the induced map f is given by

$$f(Z) = \begin{pmatrix} Z & 0 \\ k(Z) & h(Z) \end{pmatrix}$$

for some holomorphic functions k_1, k_2, h on $\Omega_{2,2}$ where $k = (k_1, k_2)$. Then by considering f on the Shilov boundary as the same method above, k should be zero. Hence f should be of the form 1 in Theorem 1.3.

Remark 4.5. Note that in generally, for one g , there could be several f . However, in case of $D_{2,2}$, $D_{3,3}$ and $\Omega_{2,2}$, $\Omega_{3,3}$, there is a unique f for each g since the number of equations and the number of unknowns are same.

4.3. More examples.

Example 4.6. If the difference of dimension gets bigger, then there are infinite number of proper holomorphic maps which are not rationally equivalent up to the automorphisms. Consider the proper holomorphic maps from $D_{2,2}$ to $D_{4,4}$. As the same method, let $P_t(x) = (x_1 + x_2 - x_3 - x_4)Q_P(x)$ where $Q_{P_t}(x) = x_1 + x_2 + x_3 + x_4 - t(x_2 + x_4)$ where $0 \leq t \leq 1$. Then

$$P_t(x) = x_1^2 + (2-t)x_1x_2 + (1-t)x_2^2 + tx_2x_3 - x_3^2 - (2-t)x_3x_4 - (1-t)x_4^2 - tx_1x_4$$

and the induced proper holomorphic maps are

$$g_t([z_1, z_2, z_3, z_4]) = [z_1^2, \sqrt{2-t}z_1z_2, \sqrt{1-t}z_2^2, \sqrt{t}z_2z_3, z_3^2, \sqrt{2-t}z_3z_4, \sqrt{1-t}z_4^2, \sqrt{t}z_1z_4].$$

This g_t satisfies the condition in Proposition 4.3 and g_t induces infinite number of proper holomorphic maps from $f_t : \Omega_{2,2} \rightarrow \Omega_{4,4}$ which is defined by (4.5)

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} z_1^2 & \sqrt{2-t}z_1z_2 & \sqrt{1-t}z_2^2 & \sqrt{t}z_2z_3 \\ \sqrt{2-t}z_1z_3 & \frac{\sqrt{2-t-t}}{\sqrt{2-t}}z_1z_4 + z_2z_3 & 2\sqrt{\frac{1-t}{2-t}}z_2z_4 & \sqrt{\frac{t}{2-t}}z_4 \\ \sqrt{1-t}z_3^2 & \frac{\sqrt{2-t-t}}{\sqrt{1-t}}z_3z_4 & z_4^2 & 0 \\ \sqrt{t}z_3 & \sqrt{t}z_4 & 0 & 0 \end{pmatrix}.$$

Remark 4.7. (2) and (3) are homotopic to each other by (4.5).

Example 4.8. There are proper holomorphic map $f : \Omega_{2,2} \rightarrow \Omega_{4,4}$ which has degree 3 polynomial in components. Let $Q_P(x) = x_1^2 + x_1x_3 + x_3^2$. Then $P(x) = x_1^3 + x_1^2x_2 + x_1x_2x_3 + x_2x_3^2 - x_3^2 - x_1^2x_4 - x_1x_3x_4 - x_3^2x_4$ and hence

$$g([x_1, x_2, x_3, x_4]) = [x_1^3, x_1^2x_2, x_1x_2x_3, x_2x_3^2, x_3^2, x_1^2x_4, x_1x_3x_4, x_3^2x_4].$$

The corresponding proper holomorphic map $f : \Omega_{2,2} \rightarrow \Omega_{4,4}$ is

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} z_1^3 & z_2 & z_1z_2 & z_1^2z_2 \\ z_1^2z_3^2 & z_4 & z_2z_3 & z_1z_2z_3 - z_1^2z_4 + z_1^2z_3z_4 \\ 3z_1z_3 - 2z_1z_3^2 & 0 & 0 & z_2z_3 + 2z_1z_4 - 2z_1z_3z_4 \\ z_3^2 & 0 & 0 & z_3z_4 \end{pmatrix}.$$

Example 4.9 (Generalized Whitney map). Consider

$$P(z) = (x_1 + \cdots + x_r - x_{r+1} - \cdots - x_{r+s})(x_1 + x_{r+1}).$$

This polynomial induces the proper meromorphic map $g : D_{r,s} \rightarrow D_{2r-1,2s-1}$ defined by

$$g([z_1, \dots, z_r, w_1, \dots, w_s]) = [z_1^2, z_1z_2, \dots, z_1z_r, w_1z_2, \dots, w_1z_r, w_1^2, w_1w_2, \dots, w_1w_s, z_1w_2, \dots, z_1w_s].$$

g induces the proper holomorphic map $f^w : \Omega_{r,s} \rightarrow \Omega_{2r-1,2s-1}$ defined by

$$(4.6) \quad \begin{pmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{pmatrix} \mapsto \begin{pmatrix} z_{11}^2 & z_{11}z_{12} & \cdots & z_{11}z_{1s} & z_{12} & \cdots & z_{1s} \\ z_{21}z_{11} & z_{21}z_{12} & \cdots & z_{21}z_{1r} & z_{22} & \cdots & z_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{r1}z_{11} & z_{r1}z_{12} & \cdots & z_{r1}z_{1s} & z_{r2} & \cdots & z_{rs} \\ z_{21} & z_{22} & \cdots & z_{2s} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \cdots & z_{rs} & 0 & \cdots & 0 \end{pmatrix}$$

This is generalized proper holomorphic map of (2) : if $r = s = 2$, f^w is same with (2) in Theorem 1.3.

Example 4.10. Consider the proper holomorphic maps from $D_{2,2}$ to $D_{3,4}$. Let $P_t(x) = (x_1 + x_2 - x_3 - x_4)Q_P(x)$ where $Q_P(x) = x_1 + tx_3$ where $0 \leq t \leq 1$. Then proper rational map $g_t : D_{2,2} \rightarrow D_{3,4}$ is given by

$$g_t([x_1, x_2, x_3, x_4]) = [x_1^2, x_1x_2, \sqrt{t}x_2x_3, \sqrt{t}x_3^2, \sqrt{t}x_3x_4, \sqrt{1-t}x_1x_3, x_1x_4].$$

The induced proper holomorphic maps $f_t : \Omega_{2,2} \rightarrow \Omega_{3,4}$ is given by

$$(4.7) \quad \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{t}z_1^2 & \sqrt{t}z_1z_2 & \sqrt{1-t}z_1 & z_2 \\ \sqrt{t}z_1z_3 & \sqrt{t}z_2z_3 & \sqrt{1-t}z_3 & z_4 \\ z_3 & z_4 & 0 & 0 \end{pmatrix}.$$

Furthermore we can generalize proper holomorphic map (4.7) to $F_t : \Omega_{r,s} \rightarrow \Omega_{2r-1,2s}$ given by for $Z = (z_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$,

$$(4.8) \quad Z \mapsto \begin{pmatrix} \sqrt{t}z_{11}^2 & \sqrt{t}z_{11}z_{12} & \cdots & \sqrt{t}z_{11}z_{1s} & \sqrt{1-t}z_{11} & z_{12} & \cdots & z_{1s} \\ \sqrt{t}z_{11}z_{21} & \sqrt{t}z_{21}z_{12} & \cdots & \sqrt{t}z_{21}z_{1r} & \sqrt{1-t}z_{21} & z_{22} & \cdots & z_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{t}z_{11}z_{r1} & \sqrt{t}z_{r1}z_{12} & \cdots & \sqrt{t}z_{r1}z_{1s} & \sqrt{1-t}z_{r1} & z_{r2} & \cdots & z_{rs} \\ z_{21} & z_{22} & \cdots & z_{2s} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \cdots & z_{rs} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Example 4.11. Consider

$$P(x) = (x_1 + \cdots + x_r - y_1 - \cdots - y_s)(x_1 + \cdots + x_r + y_1 + \cdots + y_s)$$

and the induced rational map $g : D_{r,s} \rightarrow D_{r',s'}$ where $r' = \frac{1}{2}r(r+1)$, $s' = \frac{1}{2}s(s+1)$ defined by

$$g([x_1, \dots, x_r, y_1, \dots, y_s]) = [x_1^2, \dots, x_r^2, \sqrt{2}x_1x_2, \dots, \sqrt{2}x_ix_j, \dots, \sqrt{2}x_{r-1}x_r, \\ y_1^2, \dots, y_s^2, \sqrt{2}y_1y_2, \dots, \sqrt{2}y_ky_l, \dots, \sqrt{2}y_{s-1}y_s]$$

where i, j, k and l trace over $1 \leq i < j \leq r$ and $1 \leq k < l \leq s$. Then the induced proper holomorphic map $f : \Omega_{r,s} \rightarrow \Omega_{r',s'}$ is

$$f \left(\begin{pmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{pmatrix} \right) = (M, N) \quad \text{where} \quad M = \begin{pmatrix} z_{11}^2 & \cdots & z_{1s}^2 \\ \vdots & & \vdots \\ z_{r1}^2 & \cdots & z_{rs}^2 \\ \sqrt{2}z_{11}z_{21} & \cdots & \sqrt{2}z_{1s}z_{2s} \\ \vdots & & \vdots \\ \sqrt{2}z_{i1}z_{j1} & \cdots & \sqrt{2}z_{is}z_{js} \\ \vdots & & \vdots \\ \sqrt{2}z_{r-11}z_{r1} & \cdots & \sqrt{2}z_{r-1s}z_{rs} \end{pmatrix}$$

$$N = \begin{pmatrix} \sqrt{2}z_{11}z_{12} & \cdots & \sqrt{2}z_{1k}z_{1l} & \cdots & \sqrt{2}z_{1s-1}z_{1s} \\ \vdots & & \vdots & & \vdots \\ \sqrt{2}z_{r1}z_{r2} & \cdots & \sqrt{2}z_{rk}z_{rl} & \cdots & \sqrt{2}z_{rs-1}z_{rs} \\ z_{11}z_{22} + z_{12}z_{21} & \cdots & z_{1k}z_{2l} + z_{2k}z_{1l} & \cdots & z_{1s-1}z_{2s} + z_{2s-1}z_{1s} \\ \vdots & & \vdots & & \vdots \\ z_{i1}z_{j2} + z_{j1}z_{i2} & \cdots & z_{ik}z_{jl} + z_{jk}z_{il} & \cdots & z_{is-1}z_{js} + z_{js-1}z_{is} \\ \vdots & & \vdots & & \vdots \\ z_{r-11}z_{r2} + z_{r1}z_{r-12} & \cdots & z_{r-1k}z_{rl} + z_{rk}z_{r-1l} & \cdots & z_{r-1s-1}z_{rs} + z_{rs-1}z_{r-1s} \end{pmatrix}.$$

Here i, j, k, l trace over $1 \leq i < j \leq r$ and $1 \leq k < l \leq s$.

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